ON L-M DUALITY IN REAL BANACH SPACES

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Most results in M-structure theory are related to the following two order relations that make sense for any real Banach space E:

$x \leqslant_L y$ if and only if $\|y\| = \|x\| + \|y - x\|$

$x \leqslant_M y$ if and only if every closed ball containing 0 and y contains also x.

See [1] or [2] for details.

It is the purpose of this paper to prove a new characterization of $\leqslant_M$ by starting with the obvious remark that

$x \leqslant_M y$ if and only if $\|x + z\| \leqslant \max\ \{\|z\|, \|y + z\|\}$ for every $z \in E$.

Particularly that will allow us to explain why numerical range techniques and M-theory techniques have the same bulk of applications in the context of C^*-algebras.

THEOREM 1. *Let E be a real Banach space and let x and y be two elements of E. Then the following assertions are equivalent:*

i) $x \leqslant_M y$;

ii) $\|x + z\| \leqslant \max \{\|z\|, \|y + z\|\}$ *for every* $z \in E''$ *i.e.,* $x \leqslant_M y$ *in* E'';

iii) *For every (a certain)* w' — *dense subspace* $\mathscr{H}$ *of* E',

$$f(x) \leqslant \sup \{g(y) \mid g \in E',\ g \leqslant_L f\},\ \forall f \in \mathscr{H}.$$

Proof. i) $\Rightarrow$ ii). By the principle of local reflexivity (see [4]) for each $\varepsilon > 0$ and each $z \in E''$ there exists a z_ε in E such that

$$\|x + z\| \leqslant (1 + \varepsilon)\cdot\|x + z_\varepsilon\| \leqslant (1 + \varepsilon)\cdot\max \{\|z_\varepsilon\|, \|y + z_\varepsilon\|\} \leqslant$$

$$\leqslant \frac{1 + \varepsilon}{1 - 3\varepsilon}\cdot\max \{\|z\|, \|y + z\|\}$$

so it remains to take the infimum in the right hand over $\varepsilon > 0$.

ii) $\Rightarrow$ iii). Let $f \in \mathscr{H}$, $\varepsilon > 0$ and $n \in \mathbf{N}$ be such that $n \geqslant \|y\|$. Then we can consider the sublinear functional $p_n : \mathscr{H} \to \mathbb{R}$ given by

$$p_n(\varphi) = \inf \{n\|h\| - h(y) + n\|\varphi - h\| \mid h \in \mathscr{H}\}.$$

Since $p_n(\varphi) \leqslant n\|\varphi\|$ for every $\varphi \in \mathscr{H}$, the functional p_n is continuous and thus there exists a $z \in \mathscr{H}'$ such that

$$f(z) = p_n(f)$$

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and

$$\varphi(z) \leqslant p_n(\varphi) \text{ for every } \varphi \in \mathscr{H}.$$

The second condition yields $\|z\| \leqslant n$ and $\|z + y\| \leqslant n$ so by ii),

$$f(x) + f(z) = f(x + z) \leqslant \|f\| \cdot \|x + z\| \leqslant$$

$$\leqslant \|f\| \cdot \max\{\| \|z\|, \|y + z\|\} \leqslant n\|f\|.$$

Then

$$f(x) \leqslant n\|f\| - p_n(f) =$$

$$= n\|f\| - \inf \{n\|h\| - h(y) + n\|\varphi - h\| \,|\, h \in \mathscr{H}\} =$$

$$= \sup \{n\|f\| - n\|h\| + h(y) - n\|\varphi - h\| \,|\, h \in \mathscr{H}\}.$$

We shall prove that the last term is $< a + \varepsilon$, where $a = \sup \{g(y) \,|\, g \in E', \; g \ll_L f\}$.

In fact, if the contrary is true, it would exist a sequence $(h_n)_n \subset \mathscr{H}$ such that

(*) $$n\|f\| + h_n(y) \geqslant n(\|h_n\| + \|f - h_n\|) + a + \varepsilon$$

for all $n \geqslant \|y\|$. Consequently

(**) $$h_n(y) \geqslant a + \varepsilon \text{ for } n \geqslant \|y\|,$$

which yields

$$\overline{\lim_{n\to\infty}} \|h_n\| \leqslant \lim_{n\to\infty} \frac{n\|f\| - a}{n - \|y\|} = \|f\|.$$

Particularly the sequence $(h_n)_n$ is bounded and (by passing to a subsequence if necessary) we can assume that $h_n \xrightarrow{w'} h$. By (*),

$$\|f\| + \frac{1}{n} h_n(y) \geqslant \|h_n\| + \|f - h_n\| + \frac{a + \varepsilon}{n}$$

which implies that

$$\|f\| \geqslant \|h\| + \|f - h\| \geqslant \|f\|$$

i.e., $h \ll_L f$. Then $h(y) \leqslant a$, in contradiction with (**).

iii) $\Rightarrow$ i). Let $z \in E$ and $h \in \mathscr{H}$ with $\|h\| \leqslant 1$. Then

$$h(z) + h(x) \leqslant h(z) + \sup\{g(y) \mid g \in E',\ g \ll_L h\} =$$
$$= \sup\{h(z) + g(y) \mid g \in E',\ g \ll_L h\} =$$
$$= \sup\{(h - g)(z) + g(z + y) \mid g \in E',\ g \ll_L h\} \leqslant$$
$$\leqslant \sup\{\|h - g\| \cdot \|z\| + \|g\| \cdot \|z + y\| \mid g \ll_L h\} \leqslant$$
$$\leqslant \max\{\|z\|, \|z + y\|\}$$

and thus $\|x + z\| \leqslant \max\{\|z\|, \|y + z\|\}$ for every $z \in E$ i.e., $x \ll_M y$. ■

COROLLARY 1. *Let E be a real Banach space. Then:*

i) *$x \ll_M y$ in E if and only if $f(x) \leqslant \sup\{g(y) \mid g \in E', g \ll_L f\}$ for every $f \in E'$.*

ii) *$f \ll_M g$ in E' if and only if $f(x) \leqslant \sup\{g(y) \mid y \in E'', y \ll_L x\}$ for every $x \in E$.*

From Corollary 1 ii) we can infer immediately that all $\ll_M$-intervals $[0, f]$ in E' are w'-compact and that the adjoint of every operator in the Cunningham algebra of E belongs to the centralizer of E'.

Let $\ll$ be one of the order relations $\ll_L$ and $\ll_M$. We shall say that $\ll$ is *trivial* provided that

$$x \ll y \text{ if and only if } x = \alpha \cdot y \text{ for some } \alpha \in [0, 1].$$

For example, $\ll_L$ is trivial on any strictly convex Banach space.

COROLLARY 2. *If $\ll_L$ is trivial on E' then $\ll_M$ is trivial on E.*

The duality outlined in Corollary 1 is only one way and an interesting open question is whether the assertion ii) in Corollary 1 could be straighten up to

$$f \ll_M g \text{ in } E' \text{ if and only if } f(x) \leqslant \sup\{g(y) \mid y \in E,\ y \ll_L x\}$$
$$\text{for every } x \in E.$$

A positive answer to that question would yield the following counterpart of Corollary 2: *If $\ll_L$ is trivial on E then $\ll_M$ is trivial on E'.*

How thin can be the subsets $\mathscr{H}$ as in Theorem 1 above? Alfsen and Effros have noticed in [1] that

(M) $x \ll_M y$ in E if and only if either $0 \leqslant f(x) \leqslant f(y)$ or $f(y) \leqslant f(x) \leqslant 0$ for every extreme point f of the closed unit ballk of E' (i.e., $f(x) = \alpha \cdot f(y)$ for a suitable $\alpha \in [0, 1]$).

Their argument depends upon Choquet's theory. We can offer a (somewhat) simpler argument via Corollary 1 above.

Suppose that $x \ll_M y$ in E. If f is an extreme point of K, then $g \ll_L f$ in E' yields $g = \alpha \cdot f$ for a suitable $\alpha \in [0, 1]$. See [1], p. 106. Then by Corollary 1 i),

$$f(x) \leqslant \sup \{\alpha \cdot f(y) \mid \alpha \in [0, 1]\}.$$

Since $-f$ is also an extreme point, we can restrict ourselves to the case where $f(x) > 0$. Then the inequality above yields $0 < f(x) \leqslant f(y)$.

Conversely, let $\sum_{k=1}^{n} \lambda_k f_k$ be a convex combination of extreme points of K. By hypotheses, for each k there exists an $\alpha_k \in [0, 1]$ such that $f_k(x) = \alpha_k \cdot f_k(y)$. Suppose that $\bar{B}_r(z)$ is a closed ball in E containing 0 and y. Then

$$\left| \sum_{k=1}^{n} \lambda_k f_k(x - z) \right| \leqslant \sum_{k=1}^{n} \lambda_k \alpha_k \, |f_k(y - z)| +$$

$$+ \sum_{k=1}^{n} \lambda_k (1 - \alpha_k) \, |f_k(z)| \leqslant r$$

so by Krein-Milman Theorem we can conclude that $x \in \bar{B}_r(z)$ too. ■

THEOREM 2. *Suppose that E is an M-ideal of E''. Then $x \ll_M y$ in E'' if and only if and only if for each extreme point f of the closed unit ball K of E' either $0 \leqslant f(x) \leqslant f(y)$ or $f(y) \leqslant f(x) \leqslant 0$.*

Proof. We can proceed as in the case of the assertion (M), by noticing the fact that every extreme point of K extends uniquely to an extreme point of the closed unit ball K^{00} of E'''. In fact, since E is an M-ideal of E'', then every functional $f \in E'$ has a unique extension $g \in E'''$ such that $\|g\| = \|f\|$. See [2], p. 35. ■

Theorem 2 reveals an interesting connection between $\ll_M$ and numerical range in the case of C^*-algebras. To avoid some technicalities, we shall restrict ourselves to a special case.

Let H be a complex Hilbert space and let $\mathscr{A}(H)$ be the self-adjoint part of $L(H, H)$. It is known that $\mathscr{A}(H)$ is the second dual of the Banach space E of all self-adjoint compact operators on H and the dual of the Banach space of all self-adjoint nuclear operators on H; the natural pairing of E' and E'' is given by $(A, B) \to \operatorname{Trace} AB$. See [3] for details. The extreme points of the closed unit ball of E' are of the form $\langle , x \rangle x$, where x runs over unit sphere of H. E is an M-ideal of E'' by [1], p. 167. Then by Theorem 2 above we can conclude that

$A \ll_M B$ in $\mathscr{A}(H)$ if and only if for each $x \in H$ there exists an $\alpha \in [0, 1]$ such that $\langle Ax, x \rangle = \alpha \cdot \langle Bx, x \rangle$.

Some comments are in order. Let $A, B \in \mathscr{A}(H)$.

If $B \geqslant 0$, then $A \ll_M B$ if and only if $0 \leqslant A \leqslant B$.

If $AB = BA$, then $A \ll_M B$ if and only if $A^- \leqslant B^-$ and $A^+ \leqslant B^+$. particularly, $-A^-, A^+ \ll_M A$.

If $A \ll_M B$, then $C^*AC \ll_M C^*BC$ for every $C \in L(H, H)$.

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